Boundary-layer growth near a rear stagnation point

By IAN PROUDMAN AND KATHLEEN JOHNSON
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

(Received 18 May 1961)

This paper analyses the development, according to the Navier–Stokes equations, of the two-dimensional flow in the neighbourhood of the rear stagnation point on a cylinder which is set in motion impulsively with constant velocity. The local flow is idealized to the extent that the cylindrical boundary is taken to be an infinite plane bounding a semi-infinite domain of fluid. The velocity field is taken to be a linear function of the co-ordinate measured parallel to the boundary, and the initial flow is taken to be the (unique) irrotational form of such a field, namely, inviscid flow away from a stagnation point. Thereafter this irrotational flow is maintained as the outer boundary condition at a large distance from the boundary.

It is suggested that, outside a viscous layer on the boundary, the asymptotic flow for large times is described by a similarity solution of the inviscid form of the governing equation, with a length scale normal to the boundary which increases exponentially with time. This inviscid solution has a steady velocity of slip along the boundary which is equal but opposite to that of the initial flow, so that the flow in the viscous layer ultimately becomes the well-known stagnation flow *towards* a boundary. The suggestion is supported by a numerical solution of the initial-value problem.

1. Introduction

It has been known for many years that there are no solutions of the laminar boundary-layer equations representing steady flow near the rear stagnation point on a cylinder which is held at rest in a uniform stream. This result has, with some justification, been taken as further evidence that the steady flow past cylinders for which the classical theory predicts a rear stagnation point must exhibit the phenomenon of separation, and that the whole idea of a stagnation point at which fluid leaves the cylinder is not a sensible one in a fluid of small viscosity.

During the development of a flow from rest, however, the situation is significantly different. If we take, for simplicity, the case in which the cylinder is suddenly given a constant velocity at some initial instant, then the initial flow is necessarily the classical irrotational one. In the early stages of the motion, therefore, the flow outside the boundary layer does have a rear stagnation point and it seems worth while to enquire how the flow in the neighbourhood of this point will develop in time.

The appropriate idealization appears to be as follows. We consider a semi-infinite domain of viscous incompressible fluid bounded by an infinite plane

$\bar{y} = 0$, the whole system being initially at rest. At time $\bar{t} = 0$ the fluid is instantaneously set in motion described by the stream function

$$\bar{\psi} = -\alpha \bar{x}\bar{y}, \tag{1}$$

where α is a positive rate of strain and the fixed origin of co-ordinates $(\bar{x}, \bar{y})$ is in the fixed-plane boundary. For subsequent times the irrotational flow (1) is maintained at a great distance from the boundary.

Since the development of the flow is determined by only two parameters, α and ν (= kinematic viscosity), the scales of length and time are unique and the relevant non-dimensional variables are

$$\left.\begin{aligned} \psi &= \bar{\psi}/\nu, & t &= \alpha \bar{t}, \\ x &= \alpha^{\frac{1}{2}}\bar{x}/\nu^{\frac{1}{2}}, & y &= \alpha^{\frac{1}{2}}\bar{y}/\nu^{\frac{1}{2}}. \end{aligned}\right\} \tag{2}$$

If we now make the assumption, consistent with the initial and boundary conditions, that ψ is proportional to x for all y and t, we have

$$\psi = -xF(y,t) \tag{3}$$

and, for the non-dimensional components of velocity,

$$u = \frac{\partial \psi}{\partial y} = -xF_y, \quad v = -\frac{\partial \psi}{\partial x} = F. \tag{4}$$

Thus the equations of motion yield a differential equation for F (Schlichting 1960, p. 79)

$$F_{yt} - F_y^2 + FF_{yy} - F_{yyy} = \text{function of } t \text{ only}.$$

The initial and boundary conditions are

$$\left.\begin{aligned} &F(y,0) \equiv y \quad (y \neq 0), \\ &F(0,t) = F_y(0,t) = 0 \quad (t \neq 0), \\ &F(y,t) \sim y \quad \text{as} \quad y \to \infty, \end{aligned}\right\} \tag{5}$$

the last of which reduces the above differential equation to the form

$$F_{yt} - F_y^2 + FF_{yy} - F_{yyy} = -1. \tag{6}$$

When t is small the solution may be obtained by the method developed by Blasius (1908). Thus the early stages of the diffusion of the initial vortex sheet at $y = 0$ are described by the dominant terms in (6), namely,

$$F_{yt} - F_{yyy} = 0, \tag{7}$$

and the solution satisfying the boundary conditions must, on dimensional grounds, be of the form

$$F = t^{\frac{1}{2}} \times \text{function of } (y/t^{\frac{1}{2}}). \tag{8}$$

For small values of t, therefore, the variable $\eta = y/t^{\frac{1}{2}}$ is more appropriate than y itself, and when one takes into account the relatively small inertia terms in (6), by a straightforward iteration, one finds that an expansion of the form

$$F \sim \sum_0^\infty t^{n+\frac{1}{2}} f_n(\eta) \tag{9}$$

is required. This is the structure of the Blasius series solution in the present problem.

The detailed solutions for the $f_n(\eta)$ need not concern us at this stage. It is sufficient to note that the first few functions have been found explicitly in a more general context (Blasius 1908; Goldstein & Rosenhead 1936). According to the partial sum of these first few terms, the stress on the boundary vanishes at a certain $O(1)$ value of t, and subsequently changes sign. Thus the general features of the predicted streamline pattern at times a little later than this particular

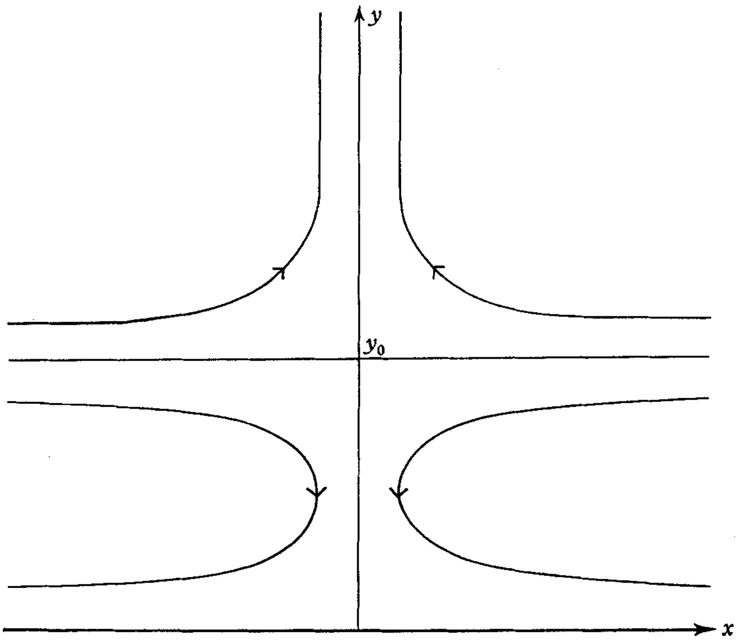

FIGURE 1. The nature of the streamlines after the onset of reversed flow.

instant are those of the accompanying sketch (figure 1). Inasmuch as the predicted time at which the flow reverses near the boundary is much greater than those for which the asymptotic expansion (9) is known to be valid, there is no real assurance that the phenomenon actually occurs. But the theory seems to be in general agreement with observations and there is little reason to doubt that it is qualitatively correct.

At this stage it is necessary to emphasize an important point which seems to have been overlooked in the literature. It has been usual to identify the phenomenon of reversed flow with boundary-layer separation and to assume that, once this has occurred, the outer boundary condition must be affected and that the whole problem becomes conceptually unsound. Whether or not one assigns the name separation to the phenomenon is, of course, purely a matter of taste. But there is no justification whatever for supposing that the outer boundary condition is affected. The initial-value problem presumably has a well-behaved solution for all finite values of t, and there is no reason to doubt the applicability of the solution to the physical flow. In particular, no trouble arises from the idealization of the flow near the rear stagnation point. The natural length scale in (2) involves the viscosity. Thus, for any fixed value of t, however large, the thickness of the rotational domain of the flow becomes arbitrarily small compared with the length scale of the cylinder as $\nu \to 0$. Moreover, the non-dimensional time t does not involve the viscosity, so we must conclude that separation, in the less trivial sense of a substantial modification of the external flow, cannot 'begin' (in the limit $\nu \to 0$) at any finite time.

Appreciation of the point discussed in the preceding paragraph raises the interesting question of the nature of the solution of (6) for large values of t. A suggested form for this asymptotic behaviour, with supporting numerical evidence, is the central contribution of the present note.

2. The asymptotic solution for large t

By the time the state sketched in figure 1 has been reached, the length scale normal to the boundary (measured by y_0, say; see figure 1) is $O(1)$ and viscous and inertia forces are of comparable importance. For subsequent times, one would expect this length scale to increase at an ever increasing rate, under the action of the convection field, thus rendering the viscous forces less and less important over most of the flow. It seems likely therefore, that most of the asymptotic flow is governed by the inviscid equation, and, further, that there are sufficiently few parameters involved for a similarity solution to be relevant.

We therefore look for a solution of the inviscid equation

$$F_{yt} - F_y^2 + FF_{yy} = -1 \tag{10}$$

in the form

$$F(y,t) = \lambda(t) f(\eta), \quad \eta = y/\lambda(t). \tag{11}$$

Substituting (11) in (10) we get

$$-\frac{\dot{\lambda}}{\lambda}\eta f'' - f'^2 + ff'' = -1, \tag{12}$$

so that

$$\frac{\dot{\lambda}}{\lambda} = \text{constant} = k$$

or*

$$\lambda = e^{kt}. \tag{13}$$

Equation (12) now becomes

$$(f - k\eta) f'' - f'^2 = -1, \tag{14}$$

of which the general first integral is

$$c^2(k\eta - f)^2 = (1 - f'^2)\left(\frac{1+f'}{1-f'}\right)^k, \tag{15}$$

where c is a constant of integration. The boundary conditions (5) give $f \sim \eta$ as $\eta \to \infty$, so that we must have $k \geqslant 1$. Moreover, in (x, η)-space, the whole of the region of (x, y)-space represented by finite values of y collapses to a thin layer on the boundary, so that the kinematic boundary condition on the normal component of velocity may be applied to the solutions of (15). Thus $f(0) = 0$, which gives $f'(0) = -1$ for all $k \geqslant 1$. Hence all of the flows represented by (15) for $k \geqslant 1$ are

* More strictly, $\log \lambda \sim kt$ as $t \to \infty$. The analysis is only approximate, and there is no assurance that (13) is free from an infinite multiplicative error as $t \to \infty$.

unsteady rotational transitions between a steady rearward stagnation flow at infinity and a steady forward stagnation flow near the boundary. In the space of the original co-ordinates (x, y), this steady forward stagnation flow near the boundary becomes the outer boundary condition at infinity, and we may expect an equilibrium boundary layer to develop on the boundary.

It does not seem to be easy to establish definitely the appropriate value of k on simple analytical grounds, though it is highly probable that the proper choice is $k = 1$. For several reasons, the solution for $k = 1$ is distinct from all the others. Most important, perhaps, is the fact that this is the only case that yields an exponential decay of vorticity away from the boundary; a condition which is commonly used (though with less justification here) to render solutions unique.

Assuming that the result $k = 1$ is correct, we may easily complete the integration to find that

$$f = \eta - \frac{2}{c}(1 - e^{-c\eta}), \tag{16}$$

where c is an arbitrary constant, presumably determined by the early development of the flow. The solution (16) is not uniformly valid near $\eta = 0$. For fixed large values of y as $t \to \infty$, (16) yields the steady flow

$$F \sim -y,$$

which becomes the outer boundary condition for the viscous flow near the boundary. Thus the asymptotic solution for all finite values of y is the well-known steady forward stagnation flow found by Hiemenz (1911).

In the following section, an approximate numerical solution of the full initial-value problem is presented. This numerical work was undertaken partly to test the general idea of an asymptotic similarity solution, partly to test the chosen value of the parameter k, and partly to determine the unknown constant c in (16).

3. The numerical solution

Equation (6) was solved in the form

$$h_t = h_{yy} + 2h - h^2 - h_y \int_0^y [1-h]\,dy, \quad \text{where} \quad h = 1 - F_y, \tag{17}$$

with the appropriate boundary conditions,

$$h(0,t) = 1 \quad (t \neq 0)$$
$$h(y,t) \sim 0 \quad \text{as} \quad y \to \infty.$$

As there is a singularity in the initial conditions at $y = 0$, the integration was started at $t = 0{\cdot}0025$ and the initial values of h were calculated from the first two terms of the Blasius series solution (9).

Several difference formulae for solving the diffusion equation are known to be stable to small numerical errors. One of these,

$$\begin{aligned} h_i(t+\Delta t) = h_i(t) + \frac{\Delta t}{\Delta y^2}(h_{i+1}(t) - 2h_i(t) + h_{i-1}(t)) + \Delta t(2h_i(t) - h_i(t)^2) \\ - \frac{\Delta t}{2\Delta y}(h_{i+1}(t) - h_{i-1}(t)) \int_0^{i\Delta y} (1 - h(y,t))\,dy, \end{aligned} \tag{18}$$

where $h_i(t)$ is the value of $h(y,t)$ at $(i\Delta y, t)$, was taken as the basis for the difference equation used to solve (17).

For the stability of the diffusion difference equation, the inequality $\Delta t/\Delta y^2 \leqslant \frac{1}{2}$ must be satisfied. Another condition on Δy and Δt is obtained by considering the flow at a large distance from the boundary. If this is described by (15), then the disturbance to the initial potential flow increases exponentially with time. To allow such an expansion to take place in the region where $h = o(1)$, the relation $\Delta y \gg e^{kt}\Delta t$ must be satisfied there.

The asymptotic condition $h \sim 0$ as $y \to \infty$ was applied by taking any value of $h < 10^{-9}$ equal to zero.

In the first stage of the solution the ratio $\Delta t/\Delta y^2$ was taken to be $\frac{1}{2}$ and the increments Δy and Δt were increased at suitable times to match the expanding length scale. Near the time of onset of reversed flow a new grid system was introduced to allow for the rapid change in h near the boundary and the increasing length scale elsewhere. This was achieved by taking Δy small near the boundary and doubling Δy at suitable values of y, whilst keeping Δt constant for the rest of the integration to $t = 5$.

A grid size was chosen that gave the smallest errors compatible with a reasonable computing time, when the difference equation (18) with the diffusion term neglected was used to integrate the similarity solution (16).

The results of the numerical integration of the full equation (17) are presented in figure 2.

From (16) we have

$$\log h = -cye^{-t} + \log 2, \tag{19}$$

and we see that if the similarity solution holds, then the graph of $\log h$ against ye^{-t} should give a straight line of gradient c. In figure 2, the values of $\log h$ are plotted against $ye^{-(t-3\cdot5)}$ for $t = 3\cdot5$, 4, 4·5 and 5. The departures from the parallel straight lines are greater for the smaller values of y, which is to be expected. The results which would be obtained for $t = 5$ if $k = 0\cdot9$ or $k = 1\cdot1$ are indicated by the broken lines.

The reason why the lines for different values of t are not coincident can be seen by appealing to the second term of the asymptotic expansion for large t. One solution involves fractional powers of ye^{-t} near $y = 0$ and seems unlikely to be relevant. Neglecting this solution we obtain

$$h = 2e^{-cye^{-t}} + Ae^{-t}e^{-cye^{-t}}$$

which gives

$$\log h = -cye^{-t} + \log(2 + Ae^{-t}), \tag{20}$$

where A is a constant to be determined by matching the solution to the forward stagnation flow near $y = 0$. For the larger values of t the intercept approaches $\log 2$, which agrees with the numerical results.

Near the boundary, the solution agrees remarkably well for the larger values of t with the known solution for steady forward stagnation flow, thus confirming the predictions of the analytical solution.

The value of c calculated from the gradient of the straight lines in figure 2 is $c = 3\cdot8$.

Thus the numerical solution gives reasonably clear evidence that the similarity olution (16) is the rele vant asymptotic solution for large t.

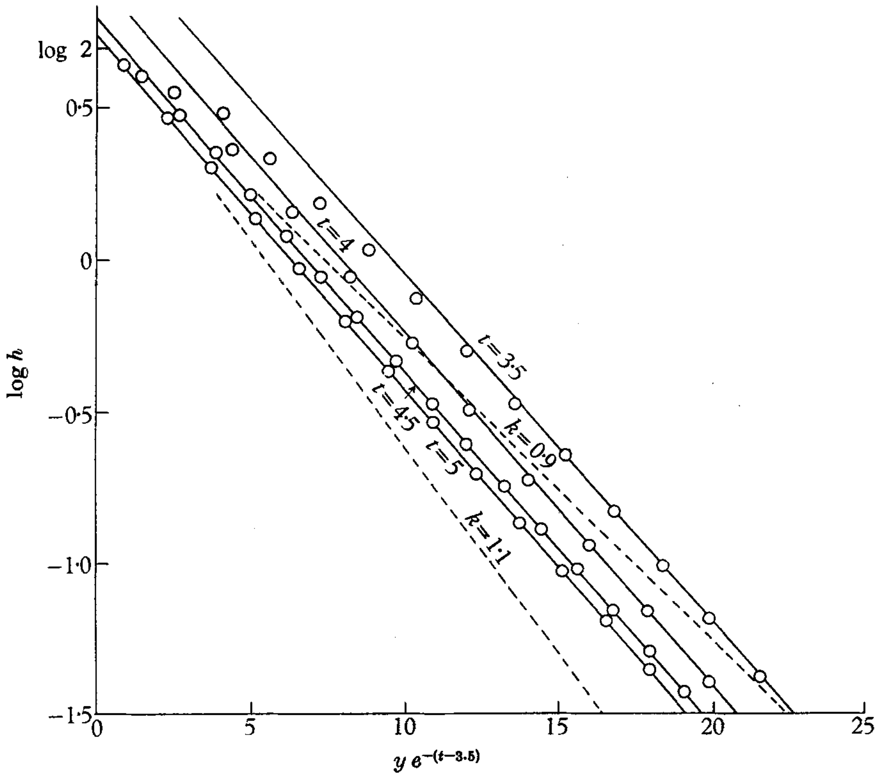

FIGURE 2. The numerical $(h, y\,e^{-(t-3\cdot5)})$ relations for $t = 3\cdot5$, 4, 4·5 and 5.

4. Some conclusions concerning the starting process

Perhaps the most interesting implication of the solution considered in the preceding sections concerns the rate of growth of eddies behind a cylinder in a stream. According to the present view, separation of the main flow past a cylinder cannot begin at any finite time in the limit as $\nu \to 0$. Yet observational evidence (see, for instance, the famous sequence of photographs due to Rubach (1914), and subsequently reproduced in many standard texts including Goldstein (1938)) apparently tends to conflict with this view. For the eddies behind a circular cylinder are observed to grow to a size comparable with that of the cylinder in the time that the cylinder takes to travel a few diameters, which strongly suggests an inviscid time-scale (in agreement with the Blasius theory) for the development of substantial disturbances to the initial irrotational flow outside the boundary layers.

However, the solution (16) provides a simple explanation of the apparent discrepancy. For a cylinder of radius a set in motion with velocity U, we have

$$\alpha = O(U/a),$$

so that (13) gives, for the width of the layer of appreciable rotational flow

$$\bar{l} = O\left(\frac{\nu^{\frac{1}{2}}a^{\frac{1}{2}}}{U^{\frac{1}{2}}}e^{U\bar{t}/a}\right). \tag{21}$$

Hence $\bar{l}$ is of order a, representing substantial disturbance of the irrotational flow, when

$$\bar{t} = O\left(\frac{a}{U}\log\frac{Ua}{\nu}\right). \tag{22}$$

Thus the relevant time scale is *almost* inviscid, depending as it does on only the logarithm of the Reynolds number. Indeed, at the Reynolds numbers for which it is possible to maintain reasonably laminar flow, even during the starting process alone, (22) would yield a very modest multiple of a/U.

We should perhaps add here that there is probably no particular instant at which separation of the main flow begins. The only special instant during the starting process is the one estimated by Blasius, and later by Goldstein and Rosenhead, for the start of reversed flow in the boundary layer. The numerical solution of § 3 estimates this time as $t = 0{\cdot}64$, which agrees, to this order of accuracy, with the estimate obtained by Goldstein & Rosenhead (1936).

A final point worth mentioning concerns the viscous shear layers that bound the separated flow region in the final steady flow past a cylinder. It is sometimes suggested that these layers are shed from the back of the cylinder, under the influence of the overall convection field, during the starting process. The solution (16) shows that this is not so, at least as far as the local mechanics is concerned. The convection field seems to be curiously inefficient in removing intense vorticity from the boundary. Apparently any fluid particle with intense vorticity is quickly forced to move towards, rather than away from, the boundary. Thus, even in a strictly inviscid fluid, an initial layer of intense vorticity near the boundary would remain there for all time.

REFERENCES

Blasius, H. 1908 *Zeit. Math. Phys.* **56**, 20.

Goldstein, S. (Ed.) 1938 *Modern Developments in Fluid Dynamics*, Vol. 1. Oxford University Press.

Goldstein, S. & Rosenhead, L. 1936 *Proc. Camb. Phil. Soc.* **32**, 392.

Hiemenz, K. 1911 *Dingl. Polytech. J.* **326**, 321.

Rubach, H. 1914 *Göttingen Dissertation.*